Teletraffic Theory and Engineering

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Contents

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1 Transfer Methods

1.1 Probability Generating Function (GF)

1.1.1 Definition of the GF

$$\boxed{G(z) = \sum_{i=0}^{\infty} p_i \cdot z^i} \tag{1}$$

$G(z)$ is called the "Probability Generating Function" (GF), as the probabilities p_i of the probability distribution can be derived from $G(z)$. The transform equation (1) means, that the whole distribution p_i is transformed in an analytic function $G(z)$ of the complex variable z.

Theorem: $G(z)$ exists in $|z| \leq 1$

1.1.2 Operations on RV´s

- Moments
  - Zeroth Moment
  $$\boxed{G(1) = 1} \tag{2}$$
  - First Moment
  $$\boxed{\mathrm{E[X]} = G'(z)|_{z=1}} \tag{3}$$
  - Variance
  $$\boxed{\mathrm{VAR[X]} = \left[\frac{d^2}{dz^2}G(z) + \frac{d}{dz}G(z) - \left(\frac{d}{dz}G(z)\right)^2\right]_{z=1}} \tag{4}$$
- Probability Distribution
$$\boxed{p_i = \frac{1}{i!} \cdot \frac{d^i}{dz^i}G(z)|_{z=0}} \tag{5}$$
- Sum of two independent Discrete RV´s
$$\boxed{G(z) = G_1(z) \cdot G_2(z)} \tag{6}$$

1.2 Table of Probability Generating Functions

Name	PDF	GF	E[X]	VAR[X]
Bernoulli Distribution	$p_i = \begin{cases} 1-q & i=0 \\ q & i=1 \\ 0 & \text{otherwise} \end{cases}$	$G(z) = (1-q) + q \cdot z$	q	$q^2 - q$
Geometric Distribution	$p_i = (1-q)^i \cdot q$	$G(z) = \frac{q}{1-(1-q)\cdot z}$	$\frac{1-q}{q}$	$\frac{1-q}{q^2}$
Binomial Distribution	$p_i = \binom{n}{i} \cdot q^i \cdot (1-q)^{n-i}$	$G(z) = [q \cdot z + (1-q)]^n$	$n \cdot q$	$n \cdot q \cdot (1-q)$
Poisson Distribution	$p_i = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$		λ	λ
Laplace Distribution	$p_i = \begin{cases} \frac{1}{N} & i = 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$		$\frac{N+1}{2}$	$\frac{N^2-1}{12}$

1.3 Laplace-Stieltjes-Transformation (LST)

1.3.1 Definition of the LST

$$\boxed{\Phi(s) = \int_{0^-}^{\infty} e^{-st} dF(t) = \int_{0^-}^{\infty} e^{-st} f(t) \cdot dt} \tag{7}$$

$\Phi(s)$ is called the LST of $F(t)$ and is also the Laplace-Transformed of $f(t)$. The transform equation (7) means that the density function $F(t)$ is transformed into an analytical function $\Phi(s)$ of the complex variable s.

Theorem: $\Phi(s)$ exists in $\Re\{s\} \geq 0$

1.3.2 Operations on RV´s

- Ordinary Moments of T

$$\boxed{\mathrm{E}[T^k] = (-1)^k \frac{d^k}{ds^k}\Phi(s)|_{s=0}} \tag{8}$$

- DF of T

$$\boxed{F(t) \overset{\mathrm{LP}}{\circ\!\!-\!\!\bullet} \frac{\Phi(s)}{s}} \tag{9}$$

- Relation between the GF and the LST for discrete RV´s

$$\boxed{\Phi(s) = G(e^{-s})} \tag{10}$$

- Sum of two independent RV´s

$$\boxed{\Phi(s) = \Phi_1(s) \cdot \Phi_2(s)} \tag{11}$$

1.3.3 Table of Laplace-Stieltjes-Transformations (LST)

Name	PDF	DF	LST	E[X]	VAR[X]
Rechteck	$p(t) = \begin{cases} \frac{1}{b-a} & a \le t \le b \\ 0 & \text{otherwise} \end{cases}$	$P(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \le t \le b \\ 1 & t > b \end{cases}$	$\frac{e^{-as}-e^{-bs}}{s\cdot(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
neg.-exponential	$p(t) = \begin{cases} \lambda \cdot e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$	$P(t) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$	$\frac{\lambda}{\lambda+s}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Deterministic	$p(t) = \delta(t - t_h)$	$P(t) = \sigma(t - t_h)$	e^{-st_h}	t_h	0

2 Stochastic Processes

2.1 Markovian Processes

Definition: A stochastic process has **Markov Property** if the future development depends only on its current state.

2.1.1 Markov Chains (MC)

- The development of the MC can be expressed by a conditional probability, called transition probability

$$\boxed{p_{ij}(t_1, t_2) = P\{X(t_2) = j | X(t_1) = i\}} \tag{1}$$

- The MC is called homogeneous, if the transition probability depends only on $(t_2 - t_1) = h$

$$\boxed{p_{ij}(h) = P\{X(t + h) = j | X(t) = i\}} \tag{2}$$

- The transition probabilities may be combined in a stochastic matrix of the transition probabilities $\underline{p}(h)$.

$$\boxed{\underline{p}(h) = \begin{bmatrix} p_{11}(h) & \cdots & p_{1n}(h) \\ \vdots & \ddots & \vdots \\ p_{n1}(h) & \cdots & p_{nn}(h) \end{bmatrix}} \tag{3}$$

  where $\forall i$ and $h \ge 0$

$$\boxed{\sum_{j=1}^{n} p_{ij}(h) = 1} \tag{4}$$

- The Chapman-Kolmogorov-Equation

$$\boxed{p_{ij}(t + h) = \sum_{\text{all } k} p_{ik}(t) \cdot p_{kj}(h) \qquad \forall i, j} \tag{5}$$

$$\boxed{\underline{p}(t + h) = \underline{p}(t) \cdot \underline{p}(h)} \tag{6}$$

– The Kolmogorov-Forward-Equation

$$\boxed{\frac{d}{dt}p_{ij}(t) = \sum_{k\neq j} q_{kj}\cdot p_{ik}(t) - q_j\cdot p_{ij}(t) \qquad \forall i,j \text{ and } t\geq 0} \tag{7}$$

– The Kolmogorov-Backward-Equation

$$\boxed{\frac{d}{dh}p_{ij}(h) = \sum_{k\neq i} q_{ik}\cdot p_{kj}(h) - q_i\cdot p_{ij}(h) \qquad \forall i,j \text{ and } h\geq 0} \tag{8}$$

– The Kolmogorov-Forward-Equation for the probability of state $P_j(t) = P\{\mathrm{X}(t) = j\}$

$$\boxed{\frac{d}{dt}P_j(t) = \sum_{k\neq j} q_{kj}\cdot P_k(t) - q_j\cdot P_j(t)} \tag{9}$$

$$\boxed{\frac{d}{dt}\underline{P}(t) = \underline{P}(t)\cdot \underline{q}} \tag{10}$$

where $\underline{q}$ is the infinitesimal generator of the stochastic matrix $\underline{p}(h)$

$$\boxed{\underline{q} = \lim_{h\to\infty}\frac{\underline{p}(h)-\mathrm{I}}{h} = \begin{bmatrix} -q_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & -q_2 & & \\ \vdots & & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & -q_n \end{bmatrix}} \tag{11}$$

- Steady-State MC
  – condition for steady-state $\forall j$

$$\boxed{P_j(t+h) = P_j(t) = p_j} \tag{12}$$

$$\boxed{\frac{d}{dt}P_j(t) = 0} \tag{13}$$

  – linear, homogeneous system of equations for steady-state probabilities (**Equilibrium Equations**):

$$\boxed{q_j p_j = \sum_{k\neq j} q_{kj} p_k} \tag{14}$$

$$\boxed{\sum_{\text{all } j} p_j = 1} \tag{15}$$

- Discrete Time MC
  – Chapman-Kolmogorov Equation:

$$\boxed{p_{ij}(m,n) = \sum_{\text{all } k} p_{ik}(m,l)\cdot p_{kj}(l,n)} \tag{16}$$

  For $l = n-1$ and for $l = m+1$ we find the Kolmogorov-Forward and the Kolmogorov-Backward Equations

  – probability of state

$$\boxed{\underline{P}(n) = \underline{P}(0)\cdot \prod_{k=1}^{n} \underline{p}(k-1,k)} \tag{17}$$

  for $\underline{p}(k-1,k) = \underline{p} \quad \forall k$ we find

$$\boxed{\underline{P}(n) = \underline{P}(0)\cdot \underline{p}^n} \tag{18}$$

  – the stationarity or steady state does not depend on n:

$$\boxed{\underline{\pi} = \lim_{n\to\infty}\underline{P}(n)} \tag{19}$$

$$\boxed{\underline{\pi} = \underline{\pi}\cdot\underline{p}} \tag{20}$$

  this set of linear homogeneous equations can be solved with the normalization equation

$$\boxed{\sum_{i=1}^{N}\pi_i = 1} \tag{21}$$

  – classification of states

$$\boxed{f_{ij}^{(n)} = P\{T_{ij} = n\}} \tag{22}$$

2.1.2 Birth and Death Processes (BDP)

Definition: A birth and death process is a special Markovian process, within which transitions occur only between direct neighbors.

- transition probability

$$\boxed{q_{xy} = \begin{cases} \lambda_x & \text{for } y = x+1 \\ \mu_x & \text{for } y = x-1 \\ 0 & \text{otherwise} \end{cases}} \tag{23}$$

  - if $\lambda_x = 0$ the BDP is called pure Death Process
  - if $\mu_x = 0$ the BDP is called pure Birth Process

- in case of stationary state

$$\boxed{p_x = \frac{\lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_{x-1}}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_x} \cdot p_0} \qquad \forall x = 1.2.\ldots, N \tag{24}$$

$$\boxed{p_0 = \left[1 + \sum_{x=1}^{N} \frac{\lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_{x-1}}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_x}\right]^{-1}} \tag{25}$$

2.2 Non-Markovian Processes

2.2.1 Renewal Processes

Definition: A renewal process is a special point process where the inter-event times T_i are stochastically independent.

2.2.2 Semi-Markovian-Processes

Definition: A stochastic process is a semi-Markovian process, if its future depends on its current state and on the duration of its current state.

2.3 Point Processes

2.3.1 Renewal Processes

- Mean Forward Recurrence Time

$$\boxed{\mathrm{E}[T^f] = \frac{1}{2} \cdot \left(\mathrm{E}[T] + \frac{\mathrm{VAR}[T]}{\mathrm{E}[T]}\right)} \tag{26}$$

- Renewal Function

$$\boxed{H(t) = \mathrm{E}[N_t]} \tag{27}$$

  and its LST

$$\boxed{H^*(s) = \frac{1}{s} \cdot \frac{\Phi(s)}{1 - \Phi(s)}} \tag{28}$$

2.3.2 Superposition and Splitting of Renewal Point Processes

- Probabilistic (Random) Splitting
  - The branch of a probabillistical splitted RPP is a RPP again.

$$\boxed{\Phi_i(s) = \frac{q_i \cdot \Phi(s)}{1 - (1 - q_i) \cdot \Phi(s)}} \tag{29}$$

  with

$$\boxed{\lambda_i = \lambda \cdot q_i} \tag{30}$$

$$\boxed{c_i^2 = q_i \cdot c^2 + (1 - q_i)} \tag{31}$$

  - Special Case: Poisson Process

$$\boxed{\Phi_i(s) = \frac{\lambda \cdot q_i}{s + \lambda \cdot q_i}} \tag{32}$$

- Deterministic Splitting

– The branch of a deterministically splitted RPP is a RPP again.

$$\boxed{\Phi_i(s) = \Phi^k(s)} \tag{33}$$

with

$$\boxed{\lambda_i = \frac{\lambda}{k}} \tag{34}$$

$$\boxed{c_i^2 = \frac{c^2}{k}} \tag{35}$$

– Special Case: Poisson Process

$$\boxed{\Phi_i(s) = \left(\frac{\lambda}{s+\lambda}\right)^k} \tag{36}$$

3 Markovian Queueing Systems

3.1 Loss and Delay Systems

3.1.1 Infinite Source Loss System M/M/n

- Probability of State

$$\boxed{p_x = \frac{\frac{A^x}{x!}}{\sum\limits_{i=0}^{n} \frac{A^i}{i!}}} \tag{1}$$

- Offered Traffic

$$\boxed{A = \lambda \cdot h = \frac{\lambda}{\epsilon}} \tag{2}$$

- Probability of Loss

$$\boxed{B = p_n = \mathrm{E}_{1,n}(A)} \tag{3}$$

→ see Appendix Ch. 4.1 Table 1 of the scriptum

- Carried Traffic

$$\boxed{Y = A \cdot (1 - B)} \tag{4}$$

→ $A \cdot B$ is the "Lost Traffic"

3.1.2 Infinite Source Delay System M/M/n

- Probabilities of State

$$\boxed{p_x = \begin{cases} p_0 \cdot \frac{A^x}{x!} & x = 0, 1, \ldots, n-1 \\ p_0 \cdot \frac{A^n}{n!} \cdot \left(\frac{A}{n}\right)^{x-n} & x = n, n+1, \ldots \end{cases}} \tag{5}$$

- Convergence Condition

$$\boxed{\frac{A}{n} \overset{!}{<} 1} \tag{6}$$

this can be rewritten as

$$\boxed{\lambda \overset{!}{<} n\epsilon} \tag{7}$$

- Probability of Delay

$$\boxed{W = p_n \cdot \frac{n}{n-A} = \mathrm{E}_{2,n}(A)} \tag{8}$$

→ see Appendix Ch. 4.1 Table 3 of the scriptum

- Occupancy

$$\boxed{\rho = \frac{A}{n}} \tag{9}$$

- Mean Queue Length

$$\boxed{\Omega = p_n \cdot \frac{\rho}{(1-\rho)^2}} \tag{10}$$

- Rate of those who have to wait is $\lambda \cdot W$

- Waiting Times
  - all requests

$$\boxed{w_1 = \frac{\Omega}{\lambda} = W \cdot t_W = W \cdot \frac{h}{n-A}} \tag{11}$$

  - waiting requests

$$\boxed{t_W = \frac{\Omega}{\lambda \cdot W} = \frac{h}{n-A}} \tag{12}$$

3.1.3 Infinite Source Delay-Loss System M/M/n with s Waiting Places

- Probability of State

$$\boxed{p_x = \begin{cases} p_0 \cdot \frac{A^x}{x!} & x = 0, 1, \ldots, n-1 \\ p_0 \cdot \frac{A^n}{n!} \cdot \left(\frac{A}{n}\right)^{x-n} & x = n, n+1, \ldots, n+s \end{cases}} \tag{13}$$

- Probability of Loss

$$\boxed{B = p_{n+s} = p_n \cdot \rho^s} \tag{14}$$

- Probability of delay

$$\boxed{W = p_n \cdot \frac{1-\rho^s}{1-\rho}} \tag{15}$$

- Mean Queue Length

$$\boxed{\Omega = p_n \cdot \rho \cdot \left[\frac{1-\rho^s}{(1-\rho)^2} - \frac{s\rho^s}{1-\rho}\right]} \tag{16}$$

- Waiting Times
  - all requests

$$\boxed{w_1 = \frac{\Omega}{\lambda}} \tag{17}$$

  - waiting requests

$$\boxed{t_W = \frac{\Omega}{\lambda \cdot W}} \tag{18}$$

- Carried Traffic

$$\boxed{Y = A(1-B)} \tag{19}$$

3.1.4 Finite Source Loss System

- Probability of State

$$\boxed{p_x = p_0 \cdot \binom{q}{x} \cdot a^x} \tag{20}$$

  with

$$\boxed{p_0 = \left[\sum_{i=0}^{n} \binom{q}{i} \cdot a^i\right]^{-1}} \tag{21}$$

$$\boxed{a = \frac{\alpha}{\epsilon}} \tag{22}$$

- Carried Traffic

$$\boxed{Y = \sum_{i=0}^{n} i \cdot p_i} \tag{23}$$

- Offered Traffic

$$\boxed{A = \lambda \cdot h = (q - Y) \cdot a} \tag{24}$$

- Time Congestion Probability

$$\boxed{E = p_n = p_0 \cdot \binom{q}{n} \cdot a^n} \tag{25}$$

- Probability of Loss

$$\boxed{B = \pi_n = \frac{q-n}{q-Y} \cdot p_n = \frac{\binom{q-1}{n} \cdot a^n}{\sum\limits_{i=0}^{n} \binom{q-1}{i} \cdot a^i}} \tag{26}$$

4 Non-Markovian Queuing Systems

4.1 Mean Value and Moment Methods

- Little's Theorem

$$\boxed{\mathrm{E}[X] = \lambda \cdot \mathrm{E}[T_F]} \tag{27}$$

- Residual Lifetime
  - PDF

$$\boxed{f_R(t) = \mu[1 - F(t)]} \tag{28}$$

  - k-th moment

$$\boxed{\mathrm{E}[T_R^k] = \frac{\mathrm{E}[T^{k+1}]}{(k+1)\mathrm{E}[T]}} \tag{29}$$

  - average

$$\boxed{\mathrm{E}[T_R] = \frac{1}{2} \cdot \left(\mathrm{E}[T] + \frac{\mathrm{VAR}[T]}{\mathrm{E}[T]}\right)} \tag{30}$$

4.2 Moment Analysis of Delay System M/G/1

- Residual Service Time

$$\boxed{\mathrm{E}[T_R] = \frac{\mathrm{E}[T_H^2]}{2 \cdot \mathrm{E}[T_H]}} \tag{31}$$

- Little's Law

$$\boxed{\Omega = \lambda \cdot w} \tag{32}$$

- Waiting Time Balance

$$\boxed{w = \rho \cdot \mathrm{E}[T_R] + \Omega \cdot h} \tag{33}$$

- Pollaczek-Khintchine-Formula

$$\boxed{w = \frac{\lambda \cdot \mathrm{E}[T_H^2]}{2 \cdot (1-\rho)} = \frac{\rho \cdot (1 + c_H^2)}{2 \cdot (1-\rho)} \cdot h} \tag{34}$$

- Comparison of Various Delay Systems

System	c_H	w
M/D/1	0	$\frac{\rho}{2 \cdot (1-\rho)} \cdot h$
M/M/1	1	$\frac{\rho}{(1-\rho)} \cdot h$
M/H$_k$/1	> 1	$\frac{\rho(1+c_H^2)}{2 \cdot (1-\rho)} \cdot h$

4.3 Delay Systems M/G/1 with Priorities

4.3.1 Average Delays in M/G/1 NP

- Class-k waiting time balance

$$\boxed{w_k = \sum_{i=1}^{P} \rho_i \cdot \mathrm{E}[T_{Ri}] + \sum_{i=1}^{k} \Omega_i \cdot h_i + \sum_{i=1}^{k-1} (w_k \cdot \lambda_i) \cdot h_i} \tag{35}$$